

# Testing instantaneous causality in presence of non constant unconditional variance

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*Abstract:* The problem of testing instantaneous causality between variables with time-varying unconditional variance is investigated. It is shown that the classical tests based on the assumption of stationary processes must be avoided in our non standard framework. More precisely we underline that the standard test does not control the type I errors, while the tests with White (1980) and Heteroscedastic Autocorrelation Consistent (HAC) corrections can suffer from a severe loss of power when the variance is not constant. Consequently a modified test based on a bootstrap procedure is proposed. The relevance of the modified test is underlined through a simulation study. The tests considered in this paper are also compared by investigating the instantaneous causality relations between US macroeconomic variables.

*Keywords:* VAR model; Unconditionally heteroscedastic errors; Instantaneous causality.

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# 1 Introduction

The concept of causality defined by Granger (1969) is widely used to analyze cause and effect relationships between macroeconomic and financial variables (see *e.g.* Sims (1972), Ashenfelter and Card (1982), Hamilton (1983), Lee (1992), Hiemstra and Jones (1994), Renault and Werker (2005), Gelper and Croux (2007)). The Granger causality has also been studied in others areas : neuroscience (see *e.g.* Brovelli et al. (2004), Seth (2008)), gene networks (Fujita et al. (2009)), geophysics (Reichel, Thejll and Lassen (2001)), or sociology (Deane and Gutmann (2003)) are some application domains among others. Causality relationships are often analyzed by taking into account only the past values of studied variables. In many situations the prediction of the unobserved current variables  $X_{2t}$  can however be improved by including the available current information of variables  $X_{1t}$ . In such a case the *instantaneous* causality relation between  $X_{1t}$  and  $X_{2t}$  is investigated (see Lütkepohl (2005, p 42)).

In the stationary VAR processes framework, the instantaneous causality is usually tested by using Wald tests for zero restrictions on the innovation's covariance matrix. Standard tools available in the commonly used softwares (see Lütkepohl and Krätzig (2004)) are based on the assumption of i.i.d. Gaussian innovations. The weight matrix of the test statistic has to be corrected by using the White type covariance matrix when the error process is assumed i.i.d. but non Gaussian (see White (1980)). In some cases models which produce nonlinear stationary processes are considered for the error terms as the GARCH or All-Pass models (see *e.g.* Bauwens, Laurent and Rombouts (2006) or Andrews, Davis and Breidt (2006)). These models allow to take into account some dependence in the innovations but also suppose

that the unconditional variance of the innovations process is constant. In order to get a standard asymptotic distribution of the Wald test statistic in these situations, Heteroscedasticity and Autocorrelation Consistent (HAC) corrections can be used (see Den Haan and Lievin (1997) for the HAC estimation).

Nevertheless many applied papers questioned the assumption of a constant unconditional variance structure. For instance Sensier and van Dijk (2004) found that most of the 214 U.S. macroeconomic variables they investigated exhibit a break in their unconditional variance. Ramey and Vine (2006) highlighted a declining variance of the U.S. automobile production and sales. Mc-Connell and Perez-Quiros (2000) documented a break in variance in the U.S. GDP growth and pointed out that neglecting non constant variance can be misleading for the data analysis. It emerges from these studies that processes with non constant unconditional variance are common features in practice. All these observations led us to consider the case of instantaneous causality relationships where the unconditional variance of the structural innovations changes over time.

Numerous tools for time series analysis in presence of non constant variance have been proposed in the literature. For instance Tsay (1988), Horváth, Kokoszka and Zhang (2006) or Sanso, Arago and Carrion (2004) proposed tests for detecting unconditional variance changes in several situations. Kokoszka and Leipus (2000) and Dahlhaus and Rao (2006) studied ARCH processes with non constant unconditional variance. Robinson (1987), Hansen (1995), Francq and Gautier (2004) or Xu and Phillips (2008) among other references investigated univariate linear models allowing for a non constant variance. Stărică (2003) considered a deterministic non constant specification for the unconditional variance of stock returns, and noted that such an approach can perform as well as the stationary GARCH(1,1) model. Kim and Park (2010) studied cointegrated systems with non constant variance. Bai (2000), Qu and Perron (2007), and Patilea and Raïssi (2012) among others investigated the estima-

tion of multivariate models with time-varying variance. Aue, Hörmann, Horvath and Reimherr (2009) proposed a test procedure for detecting variance breaks in multivariate time series.

In this paper we focus on the test of zero restriction on the time-varying variance structure. We highlight that the standard Wald test for instantaneous causality implemented in the commonly used softwares does not provide suitable critical values when the variance structure is time-varying. It is also established that the tests based on White or HAC corrections of the Wald test statistic can suffer from a severe loss power in certain important situations. More precisely these tests may be unable to detect some important alternatives as periodic changes or when the covariance structure is close to zero, so that its sign likely changes. Noting that the previous tests are not intended to handle data with non constant unconditional variance, a new approach for testing instantaneous causality taking into account non-stationary unconditional variance is proposed in this paper. It is however found that the asymptotic distribution of the modified statistic is non standard involving the unknown non constant variance structure in a functional form. When the asymptotic distribution is non standard, the wild bootstrap method is widely used in the literature for the analysis of time series possibly displaying (unconditional) heteroscedasticity/dependence (see *e.g.* Gonçalves and Kilian (2004), Horowitz, Lobato, Nankervis and Savin (2006) or Inoue and Kilian (2002)). Therefore a wild bootstrap procedure is provided for testing zero restrictions on the non constant variance structure. It is established through theoretical and empirical results that the modified test is preferable to the tests based on the spurious assumption of constant unconditional variance.

The plan of the paper is as follows. In the next section, we introduce the VAR models with non constant variance. In section 3 the testing problem for instantaneous causality between subvectors of a VAR process with non constant variance

is discussed. The asymptotic properties of the tests based on the assumption of constant unconditional variance are presented. It emerges from this part that this kind of tests should be avoided in our non standard framework. As a consequence a test based on the wild bootstrap procedure taking into consideration non constant variance is built. The finite sample properties of the tests are investigated in Section 4 by Monte Carlo experiments. We also consider US macroeconomic data to illustrate our findings. In section 5, we draw up a conclusion on our results.

## 2 Vector autoregressive model with non constant variance

Consider the following VAR model

$$X_t = A_{01}X_{t-1} + \cdots + A_{0p}X_{t-p} + u_t \quad (2.1)$$

$$u_t = H_t \epsilon_t,$$

where  $X_t \in \mathbb{R}^d$  and it is assumed that  $X_{-p+1}, \dots, X_0, X_1, \dots, X_T$  are observed. The  $d \times d$  dimensional matrices  $A_{0i}$  are such that  $\det A(z) \neq 0$  for all  $|z| \leq 1$ , where  $A(z) = I_d - \sum_{i=1}^p A_{0i} z^i$  with  $I_d$  the  $d \times d$  identity matrix. Note that the process  $(X_t)$  should be formally written in a triangular form, but the double subscript is suppressed for notational simplicity. In the following assumption we give the structure of the variance by using the rescaling approach of Dahlhaus (1997).  $\mathcal{F}_t$  corresponds to the  $\sigma$ -field generated by  $\{\epsilon_k : k \leq t\}$  and  $\|\cdot\|_r$  is such that  $\|x\|_r := (E \|x\|^r)^{1/r}$  for a random variable  $x$  with  $\|\cdot\|$  the Euclidean norm.

**Assumption A1:** (i) The  $d \times d$  matrices  $H_t$  are lower triangular nonsingular matrices with positive diagonal elements and satisfy  $H_t = G(t/T)$ , where the components of the matrix  $G(r) := \{g_{kl}(r)\}$  are measurable deterministic functions on

the interval  $(0, 1]$ , such that  $\sup_{r \in (0, 1]} |g_{kl}(r)| < \infty$ , and each  $g_{kl}$  satisfies a Lipschitz condition piecewise on a finite number of sub-intervals partitioning  $(0, 1]$ . The matrices  $\Sigma(r) = G(r)G(r)'$  are assumed positive definite for all  $r$  in  $(0, 1]$ .

(ii) The process  $(\epsilon_t)$  is  $\alpha$ -mixing and such that  $E(\epsilon_t | \mathcal{F}_{t-1}) = 0$ ,  $E(\epsilon_t \epsilon_t' | \mathcal{F}_{t-1}) = I_d$  and  $\sup_t \|\epsilon_t\|_{4\mu} < \infty$  for some  $\mu > 1$ .

If we suppose the process  $(\epsilon_t)$  Gaussian and that the functions  $g_{kl}(\cdot)$  are constant we retrieve the standard case. Nevertheless when the unconditional variance is time-varying, it can be expected that the tools developed in the stationary framework are not valid or can suffer from drawbacks since the tests for instantaneous causality are directly based on the variance structure. From the piecewise Lipschitz condition abrupt breaks as well as smooth changes are allowed for the unconditional variance. In particular the variance may have a periodic behaviour. The framework given by our assumption is similar to that of numerous papers in the literature and encompasses the case of piecewise constant variance structure (see Pesaran and Timmerman (2004), Hamori and Tokihisa (1997) or Xu and Phillips (2008) and references therein). However since we assumed that  $E(\epsilon_t \epsilon_t' | \mathcal{F}_{t-1}) = I_d$ , the error terms cannot display GARCH effects (as for instance second order correlation). Therefore the tools proposed in this paper have to be preferably used for relatively low frequency variables for which it is commonly admitted that there is no second order dynamics (for instance monthly, quarterly or annual macroeconomic data, see Section 4.2 below). In such a situation adding a multivariate GARCH structure to our model as in Hafner and Linton (2010) can be viewed as too elaborated. The tests proposed in Patilea and Raïssi (2012) can be used to check if there is no second order dynamics within the data. In the framework of **A1** we are interested in testing zero restrictions on the variance structure  $\Sigma(r)$ .

Now re-write model (2.1) as follows

$$X_t = (\tilde{X}'_{t-1} \otimes I_d)\theta_0 + u_t$$

$$u_t = H_t\epsilon_t,$$

with  $\tilde{X}_{t-1} = (X'_{t-1}, \dots, X'_{t-p})'$  and  $\theta_0 = \text{vec}\{(A_{01}, \dots, A_{0p})\}$  where  $\text{vec}(\cdot)$  is the usual column vectorization operator of a matrix and  $\otimes$  is the usual Kronecker product. The parameter vector  $\theta_0$  may be estimated by Ordinary Least Squares (OLS) or Adaptive Least Squares (ALS). Properties of these estimators are established in Patilea and Raïssi (2012) under **A1**. Denoting by  $\hat{\theta}$  the ALS (or alternatively OLS) estimator of  $\theta_0$ , we introduce the residuals  $\hat{u}_t = X_t - (\tilde{X}'_{t-1} \otimes I_d)\hat{\theta}$ . Note that the more efficient ALS estimation method should be preferred for approximating the innovations. Otherwise in practice the autoregressive order is unknown, but it is important to ensure that the lag length is well adjusted for the analysis of the instantaneous causality. For instance if the lag length is chosen too small we get correlated residuals with distorted covariance structure. On the other hand a too large autoregressive order would imply a large number of parameters to estimate in our multivariate model. The goodness-of-fit of model (2.1) can be checked by using portmanteau tests proposed in Patilea and Raïssi (2011) in our non standard framework. Hence the lag length is assumed well fitted in the sequel. More particularly the OLS and ALS estimators are  $\sqrt{T}$ -asymptotically normal if the lag length is well adjusted.

Let  $X_{1t}$  and  $X_{2t}$  be the subvectors of  $X_t := (X'_{1t}, X'_{2t})'$  with respective dimensions  $d_1$  and  $d_2$  and let  $\Sigma_t^{12}$  be the  $d_1 \times d_2$ -dimensional upper right block of  $\Sigma_t := E(u_t u_t')$ . Our goal is to determine if it exists an instantaneous causality relation between  $X_{1t}$  and  $X_{2t}$ . The next lemma gives some preliminary results and requires to introduce some additional notations. Let  $\hat{u}_t := (\hat{u}'_{1t}, \hat{u}'_{2t})'$ ,  $\hat{\vartheta}_t := \hat{u}_{2t} \otimes \hat{u}_{1t}$ ,  $\hat{v}_t := \text{vec}(\hat{u}_{1t} \hat{u}'_{2t} - \Sigma_t^{12}) = \hat{u}_{2t} \otimes \hat{u}_{1t} - \text{vec}(\Sigma_t^{12})$ ,  $H_t := (H'_{1t}, H'_{2t})'$  and  $G(r) := (G_1(r)', G_2(r)')'$  be in

line with the partition of  $X_t$ . We denote by  $[z]$  the integer part of a real number  $z$ . We also denote by  $\Rightarrow$  the convergence in distribution and  $\rightarrow$  the convergence in probability.

**Lemma 2.1.** *Under A1 we have as  $T \rightarrow \infty$*

$$T^{-1} \sum_{t=1}^T \hat{v}_t \rightarrow \int_0^1 \text{vec}(\Sigma^{12}(r)) dr, \quad (2.2)$$

and

$$T^{-\frac{1}{2}} \sum_{t=1}^T \hat{v}_t \Rightarrow \mathcal{N}(0, \Omega), \quad (2.3)$$

where

$$\Omega = \int_0^1 (G_2(r) \otimes G_1(r)) \mathcal{M} (G_2(r) \otimes G_1(r))' dr - \int_0^1 \text{vec}(\Sigma^{12}(r)) \text{vec}(\Sigma^{12}(r))' dr$$

and  $\mathcal{M} = E(\epsilon_t \epsilon_t' \otimes \epsilon_t \epsilon_t')$ . In addition we also have

$$T^{-\frac{1}{2}} \sum_{t=1}^{[Ts]} \hat{v}_t \Rightarrow \int_0^s (G_2(r) \otimes G_1(r)) dB_{\tilde{\Omega}}(r) \quad (2.4)$$

where  $B_{\tilde{\Omega}}(\cdot)$  is a Brownian Motion with covariance matrix  $\tilde{\Omega} := E(\epsilon_t \epsilon_t' \otimes \epsilon_t \epsilon_t') - \text{vec}(I_d) \text{vec}(I_d)'$ , with  $s \in [0, 1]$ .

Using (2.3) and (2.4) we shall discuss the test for instantaneous causality between  $X_{1t}$  and  $X_{2t}$  assuming spuriously that the unconditional variance is constant and propose a new test adapted to our framework in the next section. Nevertheless remarks on the result (2.3) must be made.

**Remark 2.1.** *Let be*

$$\Sigma(r) = \begin{pmatrix} \Sigma^{11}(r) & \Sigma^{12}(r) \\ \Sigma^{21}(r) & \Sigma^{22}(r) \end{pmatrix}$$



in line with the partition of  $X_t$ . If we suppose that  $\Sigma^{12}(r) = 0$  for all  $r \in (0, 1]$ , which corresponds to the case of no instantaneous causality relation between  $X_{1t}$  and  $X_{2t}$  (see the null hypothesis  $H_0$  below), it follows that  $\hat{v}_t = \hat{\vartheta}_t = \hat{u}_{2t} \otimes \hat{u}_{1t}$  and

$$\Omega = \int_0^1 (G_2(r) \otimes G_1(r)) \mathcal{M}(G_2(r) \otimes G_1(r))' dr.$$

In such a situation from Lemmas 5.1, 5.2, 5.3 and the proof of Lemma 2.1, it is clear that

$$\hat{\Omega}_w := T^{-1} \sum_{t=1}^T \hat{u}_{2t} \hat{u}_{2t}' \otimes \hat{u}_{1t} \hat{u}_{1t}' \rightarrow \Omega. \quad (2.5)$$

In particular when the process  $(u_t)$  is assumed Gaussian with non constant variance the expression of  $\Omega$  simplifies itself into  $\int_0^1 \Sigma^{22}(r) \otimes \Sigma^{11}(r) dr$  by using (5.4).

**Remark 2.2.** Assume that the process  $(u_t)$  is i.i.d. Gaussian with (constant) variance noted

$$\Sigma_u = \begin{pmatrix} \Sigma_u^{11} & 0 \\ 0 & \Sigma_u^{22} \end{pmatrix},$$

which corresponds to the standard case under the hypothesis  $H_0$  below. In such a case the expression of  $\Omega$  simplifies itself into  $\Sigma_u^{22} \otimes \Sigma_u^{11}$ . Under the strong assumption of i.i.d. Gaussian error process it can be shown that

$$\hat{\Omega}_{st} := \left\{ \left( T^{-1} \sum_{t=1}^T \hat{u}_{2t} \hat{u}_{2t}' \right) \otimes \left( T^{-1} \sum_{t=1}^T \hat{u}_{1t} \hat{u}_{1t}' \right) \right\} \rightarrow \Sigma_u^{22} \otimes \Sigma_u^{11}. \quad (2.6)$$

### 3 Testing for instantaneous causality

In the sequel we follow the notations of Lütkepohl (2005). Denote by  $X_{2t}(1|\{X_k|k < t\})$  the optimal one step linear predictor of  $X_{2t}$  at the date  $t - 1$ , based on the information of the past of the process  $(X_t)$ . Similarly we define the one step linear

predictor  $X_{2t}(1|\{X_k|k < t\} \cup \{X_{1t}\})$  based on the past of  $(X_t)$  and the present of  $(X_{1t})$ . It is said that there is no instantaneous linear causality between  $(X_{1t})$  and  $(X_{2t})$  if

$$X_{2t}(1|\{X_k|k < t\} \cup \{X_{1t}\}) = X_{2t}(1|\{X_k|k < t\}).$$

In the case of non constant variance following the assumption **A1** and, more particularly, because we assumed that the  $H_t$ 's are lower triangular nonsingular matrices with positive diagonal elements, it can be shown that there is no instantaneous causality between  $X_{1t}$  and  $X_{2t}$  if and only if the  $\Sigma_t^{12}$ 's are all equal to zero following the same arguments to those in Lütkepohl (2005, pp 46-47). Consequently in our non standard framework the following pair of hypotheses has to be tested:

$$H_0 : \Sigma^{12}(r) = 0 \text{ vs } H_1 : \Sigma^{12}(r) \neq 0 \text{ for } r \in [a, b] \subseteq [0, 1] \text{ with fixed } a < b.$$

Now if we consider the case where the variance is assumed constant  $\Sigma_t = \Sigma_u$  for all  $t$ , it is well known that there is no instantaneous causality between  $X_{2t}$  and  $X_{1t}$  if and only if  $\Sigma_u^{12} = 0$  with obvious notation. Therefore the following pair of hypotheses is tested under standard assumptions:

$$H'_0 : \Sigma_u^{12} = 0 \quad \text{vs} \quad H'_1 : \Sigma_u^{12} \neq 0,$$

The block  $\Sigma_u^{12}$  is usually estimated by  $T^{-1} \sum_{t=1}^T \hat{u}_{1t} \hat{u}'_{2t}$  which converges in probability to  $\int_0^1 \Sigma^{12}(r) dr$  under **A1**. Hence such hypothesis testing does not take into account the time-varying variance in the sense that it can only be interpreted as a *global* zero restriction testing of the covariance structure, i.e. testing  $\int_0^1 \Sigma^{12}(r) dr = 0$  against the alternative  $\int_0^1 \Sigma^{12}(r) dr \neq 0$ . Then  $H'_0$  and  $H'_1$  are inappropriate for testing instantaneous causality in our non standard framework.

It is interesting to point out that  $H_0$  is a particular case of  $H'_0$ , i.e.  $H_0 \subset H'_0$ , since  $H'_0$  corresponds to  $\int_0^1 \Sigma^{12}(r)dr = 0$ . On the other hand since  $\int_0^1 \Sigma^{12}(r)dr \neq 0$  implies that  $\Sigma^{12}(r) \neq 0$ , then  $H'_1 \subset H_1$ . More precisely, if  $\Sigma^{12}(r) \neq 0$  for  $r \in [a, b] \subseteq [0, 1]$ , we may have either  $\int_0^1 \Sigma^{12}(r)dr \neq 0$ , which corresponds to  $H_1 \cap H'_1$ , or  $\int_0^1 \Sigma^{12}(r)dr = 0$ , which corresponds to  $H_1 \cap H'_0$ . Note that we have  $H_1 = (H_1 \cap H'_1) \cup (H_1 \cap H'_0)$  and  $(H_1 \cap H'_1) \cap (H_1 \cap H'_0) = \emptyset$ . It is shown in the next part that the case  $H_1 \cap H'_0$  entails a loss of power for tests built on the assumption of constant unconditional variance of the innovations.

### 3.1 Tests based on the assumption of constant error variance

In this section the consequences of non constant variance on the instantaneous causality tests based on the spurious assumption of a stationary process are analyzed. Let be  $\delta_T := T^{-\frac{1}{2}} \sum_{t=1}^T \hat{\vartheta}_t$  where we recall that  $\hat{\vartheta}_t = \hat{u}_{2t} \otimes \hat{u}_{1t}$ . The standard test statistic is given by

$$S_{st} = \delta'_T (\hat{\Omega}_{st})^{-1} \delta_T,$$

where  $\hat{\Omega}_{st}$  is defined in (2.6). Under **A1** it can be shown that  $\hat{\Omega}_{st} \rightarrow \int_0^1 \Sigma^{22}(r)dr \otimes \int_0^1 \Sigma^{11}(r)dr =: \Omega_{st}$  and we obviously have  $\Omega \neq \Omega_{st}$  in general.

If the practitioner (spuriously) assumes that the error process is iid but not Gaussian,  $u_{1t}$  and  $u_{2t}$  could be dependent and the following statistic with White type correction should be used:

$$S_w = \delta'_T (\hat{\Omega}_w)^{-1} \delta_T,$$

where the weight matrix  $\hat{\Omega}_w$  is defined in (2.5). Recall that  $\hat{\Omega}_w$  is a consistent

estimator of  $\Omega$  under  $H_0$  and it is clear from the proof of Lemma 2.1 that

$$\hat{\Omega}_w \rightarrow \Omega + \int_0^1 \text{vec}(\Sigma^{12}(r)) \text{vec}(\Sigma^{12}(r))' dr \quad (3.1)$$

under  $H_1$ .

Finally the practitioner may again (spuriously) suppose that the error process is stationary and that the observed heteroscedasticity is a consequence of the presence of nonlinearities. However note that the assumed heteroscedasticity is only conditional while the unconditional variance is still constant in this case. This kind of situation can arise if we (spuriously) assume that the innovations process is driven by a GARCH model or any other model displaying nonlinearities such as models driven by hidden Markov chains or All-Pass models (see Amendola and Francq (2009)). Note that the test proposed in Sanso, Arago and Carrion (2004) could be used to detect changes in the unconditional variance. In such a case HAC type weight matrices should be used in the test statistic. For simplicity we focus on the VARHAC weight matrix (see Den Haan and Levin (1997)). Denote by  $\hat{\mathcal{A}}_{m,1}, \dots, \hat{\mathcal{A}}_{m,m}$  the coefficients of the LS regression of  $\hat{v}_t$  on  $\hat{v}_{t-1}, \dots, \hat{v}_{t-m}$ , taking  $\hat{v}_t = 0$  for  $t \leq 0$ . Introduce  $\hat{z}_{m,t}$  the residuals of such a regression and  $\hat{\Omega}_h = \mathcal{A}(1)^{-1} \hat{\Sigma}_z \mathcal{A}(1)^{-1}$  where  $\mathcal{A}(1) = I_{d_1 d_2} - \sum_{k=1}^m \hat{\mathcal{A}}_{m,k}$  and  $\hat{\Sigma}_z = T^{-1} \sum_{t=1}^T \hat{z}_{m,t} \hat{z}_{m,t}'$ . The order  $m$  can be chosen by using an information criterion. The following statistic involving VARHAC type weight matrix may be used

$$S_h = \delta_T' (\hat{\Omega}_h)^{-1} \delta_T.$$

Since we assumed that the autoregressive order  $p$  is well adjusted (or known), the process  $v_t = u_{2t} \otimes u_{1t}$  is uncorrelated and it can be shown that the  $\mathcal{A}_{m,k}$ 's converge to zero in probability. Therefore  $\hat{\Omega}_h \rightarrow \Omega$  under  $H_0$  and

$$\hat{\Omega}_h \rightarrow \Omega + \int_0^1 \text{vec}(\Sigma^{12}(r)) \text{vec}(\Sigma^{12}(r))' dr, \quad (3.2)$$

under  $H_1$ , so that  $\hat{\Omega}_h$  and  $\hat{\Omega}_w$  are asymptotically equivalent in the framework of **A1**. This is not surprising since second order dynamics are in fact excluded in **A1**.

In this part the asymptotic properties of the above statistics are investigated. The asymptotic behavior of the statistics in our non standard framework is first established under  $H_0$ . The results are direct consequences of (2.3)

**Proposition 1.** *Assume that  $H_0$  hold. Then under **A1** we have as  $T \rightarrow \infty$*

$$S_{st} \Rightarrow \sum_{j=1}^{d_1 d_2} \lambda_j Z_j^2, \quad (3.3)$$

where the  $Z_j$ 's are independent  $\mathcal{N}(0, 1)$  variables, and  $\lambda_1, \dots, \lambda_{d_1 d_2}$  are the eigenvalues of the matrix  $\Omega_{st}^{-\frac{1}{2}} \Omega \Omega_{st}^{-\frac{1}{2}}$ . In addition we also have

$$S_w \Rightarrow \chi_{d_1 d_2}^2 \quad \text{and} \quad S_h \Rightarrow \chi_{d_1 d_2}^2. \quad (3.4)$$

For a fixed asymptotic level  $\alpha$ , the standard test ( $W_{st}$  hereafter) consists in rejecting the hypothesis of no instantaneous causality between  $X_{1t}$  and  $X_{2t}$  if  $S_{st} > \chi_{d_1 d_2, 1-\alpha}^2$  where  $\chi_{d_1 d_2, 1-\alpha}^2$  is the  $(1 - \alpha)$ th quantile of the  $\chi_{d_1 d_2}^2$  law. Therefore it appears from (3.3) that the standard test is not able to control the type I error since  $\Omega_{st} \neq \Omega$  in general. Denote by  $W_w$  (resp.  $W_h$ ) the test consisting to reject the hypothesis of no instantaneous causality if  $S_w > \chi_{d_1 d_2, 1-\alpha}^2$  (resp.  $S_h > \chi_{d_1 d_2, 1-\alpha}^2$ ). From (3.4) we see that the  $W_w$  and  $W_h$  tests should have good size properties for large enough  $T$ .

Now we turn to the study of the power properties of the  $W_{st}$ ,  $W_w$  and  $W_h$  tests in the cases  $H_1 \cap H'_1$  and  $H_1 \cap H'_0$ . We first consider the situation  $H_1 \cap H'_1$ , which corresponds to the case  $\int_0^1 \Sigma^{12}(r) dr \neq 0$ . From (2.2) it is clear that  $T^{-1}S_i$ , with

$i = st, w, h$ , converge in probability to strictly positive constants if  $\int_0^1 \Sigma^{12}(r)dr \neq 0$ . It follows that the  $S_{st}$ ,  $S_w$  and  $S_h$  statistics grow to infinity as fast as  $T$ . Therefore we can expect that the tests based on the assumption of constant variance will detect a possible instantaneous causality for large enough sample sizes when  $H_1 \cap H'_1$  hold. The abilities of the  $W_{st}$ ,  $W_w$  and  $W_h$  tests to detect the case  $\int_0^1 \Sigma^{12}(r)dr \neq 0$  are compared considering the approximate Bahadur slope approach (Bahadur (1960)). For the test based on the  $S_{st}$  statistic define  $q_{st}(x) = P\left(\sum_{j=1}^{d_1 d_2} \lambda_j Z_j^2 > x\right)$  for any  $x > 0$ . For a fixed alternative such that  $\varpi = \int_0^1 \Sigma^{12}(r)dr \neq 0$ , consider the asymptotic slope  $c_{st}(\varpi) = 2 \lim_{T \rightarrow \infty} T^{-1} q_{st}(S_{st})$ . Define similarly  $c_w(\varpi)$  and  $c_h(\varpi)$  for the  $W_w$ ,  $W_h$  tests and also the asymptotic relative efficiencies  $ARE_{S_w, S_{st}}(\varpi) = c_w(\varpi)/c_{st}(\varpi)$  and  $ARE_{S_h, S_{st}}(\varpi) = c_h(\varpi)/c_{st}(\varpi)$ . A relative efficiency  $ARE_{S_h, S_{st}}(\varpi) \geq 1$  suggests that the  $W_h$  test is more able to detect the case  $\int_0^1 \Sigma^{12}(r)dr \neq 0$  than the test based on the  $S_{st}$  statistic. In such a case the  $p$ -values of the  $W_h$  test converge faster to zero than those of the  $W_{st}$  test.

**Proposition 2.** *Under A1 we have  $ARE_{S_w, S_{st}}(\varpi) \geq 1$  and  $ARE_{S_h, S_{st}}(\varpi) \geq 1$  for every alternative such that  $\varpi = \int_0^1 \Sigma^{12}(r)dr \neq 0$ .*

The proof of Proposition 2 is similar to the proof of Proposition 5.3 in Patilea and Raïssi (2012) and is then omitted. When the errors are assumed iid Gaussian we obtain  $ARE_{S_w, S_{st}}(\omega) = 1$  and  $ARE_{S_h, S_{st}}(\omega) = 1$ . Nevertheless if the variance structure of the errors is non constant with  $\int_0^1 \Sigma^{12}(r)dr \neq 0$ , the  $W_h$  or  $W_w$  tests achieve a gain in power when compared to the  $W_{st}$  test.

Finally we study the ability of the  $W_{st}$ ,  $W_w$  and  $W_h$  tests in detecting instantaneous causality when  $H_1 \cap H'_0$  hold, that is  $\Sigma^{12}(r) \neq 0$  and  $\int_0^1 \Sigma^{12}(r)dr = 0$ . In this case  $\hat{\Omega}_h = O_p(1)$  and  $\hat{\Omega}_w = O_p(1)$  from (3.1) and (3.2) and we also have

$\hat{\Omega}_{st} = O_p(1)$ , while the non centrality term is  $T^{-\frac{1}{2}} \sum_{t=1}^T \Sigma_t^{12} = o(T^{\frac{1}{2}})$ . Therefore we have  $S_i = o_p(T)$  with  $i = st, w, h$  in the case  $H_1 \cap H'_0$ . When such eventuality is considered, it is clear that the tests based on the assumption of stationary errors may suffer from a severe loss of power. This is a consequence of the fact that this kind of tests are not intended to take into account time varying variance. The case  $H_1 \cap H'_0$  can arise in the important case where  $\Sigma^{12}(r) \neq 0$  but close to zero so that  $\Sigma^{12}(r)$  may have a changing sign. Even when at least one of the components of  $\Sigma^{12}(r)$  is far from zero, we can have  $\int_0^1 \Sigma^{12}(r) dr = 0$  as for instance in some cases where the variance structure is periodic. This can be seen by considering the bivariate case and taking  $\Sigma^{12}(r) = c \cos(\pi r)$  or  $\Sigma^{12}(r) = c \mathbf{1}_{[0, \frac{1}{2}]}(r) - c \mathbf{1}_{] \frac{1}{2}, 1]}(r)$  with  $c \in \mathbb{R}$ . Therefore the tests based on the spurious assumption of constant unconditional variance for the error must be avoided.

In summary it is found that the tests based on the White and VARHAC corrections should control well the type I errors for large enough samples on the contrary to the  $W_{st}$ . In addition it appears that in the case of non constant unconditional variance and when  $H_1 \cap H'_1$  hold, the  $W_h$  and  $W_w$  tests have better power properties than the  $W_{st}$  test. Therefore the  $W_h$  and  $W_w$  tests should be preferred to the  $W_{st}$  test when the unconditional variance is time-varying. However it is also found that the tests based on the assumption of constant variance may suffer from a severe loss of power in the important cases where  $\int_0^1 \Sigma^{12}(r) dr = 0$  (or  $\int_0^1 \Sigma^{12}(r) dr \approx 0$ ). A bootstrap test circumventing this power problem in the case  $H_1 \cap H'_0$  is proposed in the next part.

### 3.2 A bootstrap test taking into account non constant variance

Introduce  $\delta_s = T^{-\frac{1}{2}} \sum_{t=1}^{[Ts]} \hat{v}_t$  with  $s \in [0, 1]$  and consider the following statistic:

$$S_b = \sup_{s \in [0,1]} \|\delta_s\|_2^2.$$

Under  $H_0$  and from (2.4) we write:

$$\delta_s \Rightarrow \int_0^s (G_2(r) \otimes G_1(r)) dB_{\tilde{\Omega}}(r) := K(s),$$

where the covariance matrix becomes  $\tilde{\Omega} = \mathcal{M} = E(\epsilon_t \epsilon_t' \otimes \epsilon_t \epsilon_t')$ . Therefore under  $H_0$  we have from the Continuous Mapping Theorem

$$S_b \Rightarrow \sup_{s \in [0,1]} \|K(s)\|_2^2, \quad (3.5)$$

since the functional  $f(Y) = \sup_{s \in [0,1]} \|Y(s)\|_2^2$  is continuous for any  $Y \in D[0, 1]$ , the space of càdlàg processes on  $[0,1]$ . Under  $H_1$  we obtain

$$T^{-\frac{1}{2}} \delta_s = T^{-1} \sum_{t=1}^{[Ts]} \hat{v}_t + T^{-1} \sum_{t=1}^{[Ts]} \text{vec}(\Sigma_t^{12}) \quad (3.6)$$

with  $\tilde{\Omega}$  defined in (2.4). The first term in the right hand side of (3.6) converges to zero in probability, while we have  $T^{-1} \sum_{t=1}^{[Ts]} \text{vec}(\Sigma_t^{12}) = \int_0^s \text{vec}(\Sigma^{12}(r)) dr + o(1)$  and  $\sup_{s \in [0,1]} \|\{\int_0^s \text{vec}(\Sigma^{12}(r)) dr\}\|_2^2 = C > 0$ . Hence we have in such a situation  $S_b = CT + o_p(T)$ .

From (3.5) we see that the asymptotic distribution of  $S_b$  under the null  $H_0$  is non standard and depends on the unknown variance structure and the fourth order cumulants of the process  $(\epsilon_t)$  in a functional form. Thus the statistic  $S_b$  cannot directly be used to build a test and we consider a wild bootstrap procedure to provide reliable quantiles for testing the instantaneous causality. In the literature such pro-



cedures were used for investigating VAR model specification as in Inoue and Kilian (2002) among others. The reader is referred to Davidson and Flachaire (2008) or Gonçalves and Kilian (2004, 2007) and references therein for the wild bootstrap procedure method. For resampling our test statistic we draw  $B$  bootstrap sets given by  $\vartheta_t^{(i)} := \xi_t^{(i)} \hat{\vartheta}_t = \xi_t^{(i)} \hat{u}_{2t} \otimes \hat{u}_{1t}$ ,  $t \in \{1, \dots, T\}$  and  $i \in \{1, \dots, B\}$ , where the univariate random variables  $\xi_t^{(i)}$  are taken iid standard Gaussian, independent from  $(u_t)$ . For a given  $i \in \{1, \dots, B\}$  set  $\delta_s^{(i)} = T^{-\frac{1}{2}} \sum_{t=1}^{[Ts]} \vartheta_t^{(i)}$  and  $S_b^{(i)} = \sup_{s \in [0,1]} \|\delta_s^{(i)}\|_2^2$ . In our procedure bootstrap counterparts of the  $x_t$ 's are not generated and the residuals are directly used to generate the bootstrap residuals. This is motivated by the fact that zero restrictions are tested on the variance structure, so that we only consider the residuals in the test statistic. In addition it is seen from (5.3) that the residuals and the errors are asymptotically equivalent. It is also clear that the wild bootstrap method is designed to replicate the pattern of non constant variance of the residuals in  $S_b^{(i)}$ . More precisely we have under **A1**

$$S_b^{(i)} \Rightarrow^P \sup_{s \in [0,1]} \|K(s)\|_2^2, \quad (3.7)$$

where we denote by  $\Rightarrow^P$  the weak convergence in probability. A proof of (3.7) is provided in the Appendix. Note that we have by construction  $E^*(\xi_t^{(i)} \hat{\vartheta}_t) = 0$  even when the alternative is true, that is  $E(\vartheta_t) \neq 0$  (recall that  $\vartheta_t = u_{2t} \otimes u_{1t}$ ). As a consequence the result (3.7) is hold whatever  $\Sigma(r)^{12} = 0$  or  $\Sigma(r)^{12} \neq 0$ .

The  $W_b$  test consists in rejecting  $H_0$  if the statistic  $S_b$  exceeds the  $(1 - \alpha)$  quantile of the bootstrap distribution. Under  $H_1$  with  $\int_0^1 \Sigma(r)^{12} dr \neq 0$  we note that all the statistics considered in this paper increase at the rate  $T$ . However when  $\Sigma(r)^{12} \neq 0$  with  $\int_0^1 \Sigma(r)^{12} dr \approx 0$ , we can expect that the  $W_b$  test is more powerful than the tests based on the assumption of constant unconditional variance. In such situations we may have  $S_w = o_p(T)$ ,  $S_h = o_p(T)$  while again  $S_b = O_p(T)$ . If the unconditional

variance is constant, that is  $\Sigma^{12}(r) = \Sigma_u^{12}$ , note that  $S_w = O_p(T)$ ,  $S_h = O_p(T)$  and  $S_b = O_p(T)$ . Hence we can expect no major loss of power for the  $W_b$  when compared to the  $W_w$  and  $W_h$  tests if the underlying structure of the variance is constant. In general since  $S_b = O_p(T)$  and in view of (3.7), the  $W_b$  test is consistent. From the above results we can draw the conclusion that the  $W_b$  is preferable if the unconditional variance is non constant for large enough sample sizes.

## 4 Numerical illustrations

In this section the  $W_b$  test is compared to the  $W_{st}$  and  $W_w$  tests. The VARHAC statistic being asymptotically equivalent to the White statistic under **A1**, as noted above, we did not take into account this test in our comparisons. First the type I errors and power properties of the three tests are compared using simulated bivariate VAR(1) processes with unconditional time-varying variance. The tests are next applied to two macroeconomic data sets.

### 4.1 Simulation study

For our experiments we simulated simple bivariate VAR(1) processes where the autoregressive parameters are inspired from those estimated from the money supply and inflation in the U.S. data (see section below). The data generating process can be written as

$$\begin{pmatrix} X_{1,t} \\ X_{2,t} \end{pmatrix} = \begin{pmatrix} 0.64 & -1 \\ -0.01 & 0.44 \end{pmatrix} \begin{pmatrix} X_{1,t-1} \\ X_{2,t-1} \end{pmatrix} + \begin{pmatrix} u_{1,t} \\ u_{2,t} \end{pmatrix} \quad (4.1)$$

where the innovations are Gaussian with variance structure  $\Sigma(r)$  respecting the assumption **A1**. Two cases are considered for this structure :

- **Case 1: Empirical size setting.** It does not exist an instantaneous causality

relation between  $X_{1,t}$  and  $X_{2,t}$  :

$$\Sigma(r) = \begin{pmatrix} \Sigma^{11}(r) & 0 \\ 0 & \Sigma^{22}(r) \end{pmatrix} \quad \forall r \in (0, 1].$$

where  $\Sigma^{11}(r) = a - \cos(br)$  and  $\Sigma^{22}(r) = a + \sin(br)$  correspond to the non constant variances of the innovations. We take  $a > 1$  which represents the level of these variances and  $b$  their angular frequency.

- **Case 2: Empirical power setting.** It exists an instantaneous causality relation between  $X_{1,t}$  and  $X_{2,t}$  :

$$\Sigma(r) = \begin{pmatrix} \Sigma^{11}(r) & \Sigma^{12}(r) \\ \Sigma^{12}(r) & \Sigma^{22}(r) \end{pmatrix} \quad \forall r \in (0, 1].$$

where  $\Sigma^{12}(r) = c \sin(2\pi r)$  respects the case  $\int_0^1 \Sigma^{12}(r) dr = 0$  with  $\Sigma^{12}(r) \neq 0$  almost everywhere on  $r \in (0, 1]$ , and  $\Sigma^{11}(\cdot)$ ,  $\Sigma^{22}(\cdot)$  are defined as in Case 1. In particular, the constant  $c$  will allow to investigate the ability of our modified test for detecting such alternative when it gets closer to the null hypothesis.

Note that  $a$ ,  $b$  and  $c$  have to be chosen to fulfill the positive definite condition on  $\Sigma(r)$  for all  $r$  in  $(0, 1]$ . For instance this property is checked if  $a = 1.1$ ,  $b = 11$  and  $\frac{2}{3} \geq c > 0$ .

The finite sample properties of the tests are assessed by means of the following Monte Carlo experiments. For each sample size, 1000 time series following (4.1) are generated. The lag length is assumed known and the autoregressive parameters are estimated by using the commonly used OLS method. In all our experiments we use 299 bootstrap iterations for the  $W_b$  test. We use processes generated by **Case 1** to shed light on the control of the type I errors of the studied tests. The results are reported in Table 1. On the other hand processes generated by **Case 2** are considered for the power study. The results are given in Table 2 and Figure 1. Note

that in Table 1 and Table 2 we take  $a = 1.1$ ,  $b = 11$  and  $c = 0.5$  while in Figure 1 we take several values for  $c$  and  $a = 1.1$ ,  $b = 11$ .

		Asymptotic nominal level								
		1%			5%			10%		
		$W_{st}$	$W_w$	$W_b$	$W_{st}$	$W_w$	$W_b$	$W_{st}$	$W_w$	$W_b$
Sample size	50	0.010	0.007	0.007	0.048	0.050	0.047	0.102	0.113	0.105
	100	0.009	0.009	0.010	0.057	0.066	0.066	0.095	0.096	0.107
	200	0.011	0.010	0.011	0.045	0.047	0.052	0.101	0.116	0.122
	500	0.011	0.010	0.010	0.042	0.047	0.050	0.099	0.101	0.101
	1000	0.008	0.009	0.010	0.056	0.051	0.051	0.088	0.097	0.103

Table 1: The empirical size for the studied tests with asymptotic nominal level 1%, 5%, 10% and  $a = 1.1$ ,  $b = 11$ ,  $c = 0.5$ .

		Asymptotic nominal level								
		1%			5%			10%		
		$W_{st}$	$W_w$	$W_b$	$W_{st}$	$W_w$	$W_b$	$W_{st}$	$W_w$	$W_b$
Sample size	50	0.014	0.005	0.003	0.056	0.040	0.045	0.110	0.085	0.132
	100	0.012	0.005	0.012	0.056	0.038	0.102	0.098	0.079	0.213
	200	0.017	0.010	0.076	0.063	0.048	0.305	0.106	0.093	0.512
	500	0.011	0.006	0.486	0.050	0.038	0.837	0.105	0.080	0.930
	1000	0.015	0.011	0.966	0.056	0.045	0.997	0.108	0.088	1.000

Table 2: The empirical power for the studied tests based on asymptotic nominal levels 1%, 5%, 10% and  $a = 1.1$ ,  $b = 11$ ,  $c = 0.5$ .

In our example the  $W_{st}$ ,  $W_w$  and  $W_b$  tests seem to control the type I errors reasonably well (see Table 1). We can remark that the standard test provides similar

results as compared to the other tests. Nevertheless this outcome does not have to be generalized in view of (3.3). In addition recall from Proposition 2 that the  $W_{st}$  is less powerful than the  $W_w$  and  $W_h$  tests. Now if we turn to the alternative given by **Case 2**, Table 2 clearly shows that the  $W_{st}$  and  $W_w$  tests have no power as the sample sizes increase on the contrary of the  $W_b$  test. This confirms the theoretical results obtained when  $\int_0^1 \Sigma^{12}(r)dr \approx 0$ . For instance the  $W_b$  test is almost always rejecting the null hypothesis  $H_0$  for a sample size of 1000, while the  $W_{st}$  and  $W_w$  tests are completely not able to detect the alternative in this case.

In the above power experiments the changes of  $\Sigma^{12}(r)$  around zero were fixed by a constant  $c$ . In this part we illustrate the ability of the tests to detect departures from the null hypothesis  $\Sigma^{12}(r) = 0$ , while we again have  $\int_0^1 \Sigma^{12}(r)dr = 0$  in all situations. Figure 1 represents the power of the three tests when the parameter  $c$  takes several values, while the sample is fixed  $T = 500$ . We clearly observe that the relative rejection frequencies of the  $W_b$  test increases when the covariance structure  $\Sigma^{12}(r) \neq 0$  goes away from zero but verifying  $\int_0^1 \Sigma^{12}(r)dr = 0$ . On the other hand we again remark that the relative rejection frequencies of the tests based on the assumption of constant variance remain close to the asymptotic nominal level even when  $c$  takes large values.

## 4.2 Application to macroeconomic data sets

In this part we compare the  $W_{st}$  and  $W_w$  tests with the  $W_b$  test by investigating instantaneous causality relationships in U.S. macroeconomic data sets.

### 4.2.1 Money supply and inflation in the U.S.A.

The relationship between money supply and inflation is fundamental in the macroeconomic theories explaining the influence of monetary policy on economy. For in-

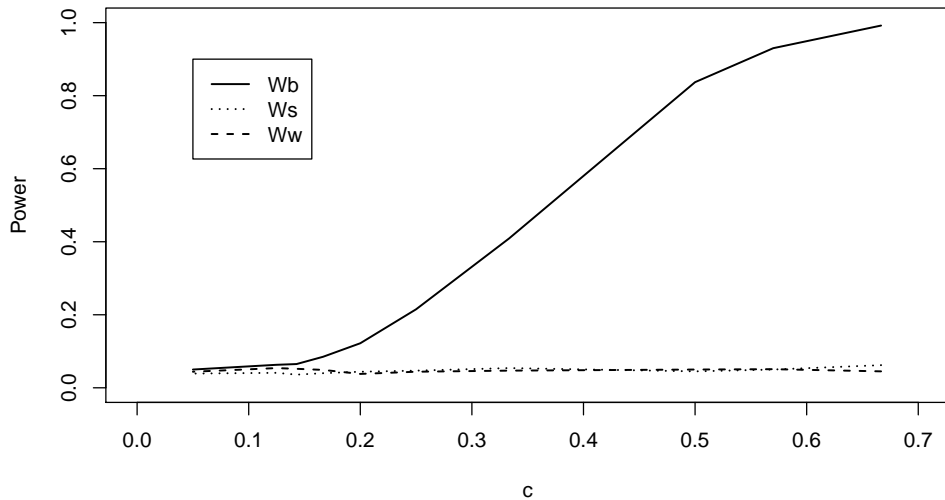


Figure 1: The empirical power of the  $W_b$ ,  $W_{st}$  and  $W_w$  tests for fixed sample size  $T = 500$  and varying  $c$  parameter. The asymptotic nominal level is 5% and we take  $a = 1.1$ ,  $b = 11$ .

stance, the quantity theory of money assumes a proportional relationship between money supply and the price level. The reader is referred to Case, Fair and Oster (2011) or Mankiw and Taylor (2006) concerning the theoretical links which can be made between money supply and inflation. Many studies investigate this relation from an empirical point of view. Their results are however ambiguous. For instance, Turnovsky and Wohar (1984) used a simple macro model to investigate the relationship and find that the rate of inflation is independent of the monetary growth rate in the U.S.A. over the period 1923-1960, while Benderly and Zwick (1985) or Jones and Uri (1986) give some evidence of relationship over the respective periods 1955-1982 and 1953-1984. Here we investigate the hypothesis of an instantaneous causal relationship between money supply and inflation in the U.S.A. over the period 1979-1995.

The data considered here are the M1 money stock (M1) and the Producer Price

Index for all commodities ( $PPI_{ACO}$ ). The M1 represents the money supply and  $PPI_{ACO}$  the inflation from the point of view of producers. The M1 index is provided by the Board of Governors of the Federal Reserve System while the  $PPI_{ACO}$  index is provided by the US Department of Labor. The data are taken from 04/1979 to 12/1995 with a monthly frequency and are available on the web site of the Federal Reserve Bank of St. Louis (Series ID : M1 and PPIACO). The length of the series is  $T = 200$ .

The first differences of the data are considered in the sequel. From Figure 2 it appears that the obtained series have non constant variance. We adjusted a VAR(1) model to the first differences of the series. The autoregressive order is chosen by using portmanteau tests adapted to our non standard framework where the variance structure  $\Sigma(r)$  is time-varying (see Patilea and Raïssi (2011) for details). The outcomes in Table 3 suggest that the model is well fitted. The estimation of the model by the OLS method is given in Table 4. The residuals of this estimation are next recovered to implement the tests studied in this paper. Note that we used 399 bootstrap iterations for the  $W_b$  test.

From Table 5 we see that the  $p$ -value of the  $W_b$  test is quite different from those of the  $W_{st}$  and  $W_w$  tests. For instance the null hypothesis of no instantaneous causality is rejected by the  $W_b$  test for a significance level of 10% while it is accepted by the other tests. These observations can be explained by the covariance structure of the innovations. Indeed the nonparametric estimation of this covariance structure plotted in Figure 3 shows that  $\Sigma^{12}(r)$  seems not null over the considered period while it seems that  $\int_0^1 \Sigma^{12}(r) \approx 0$ .

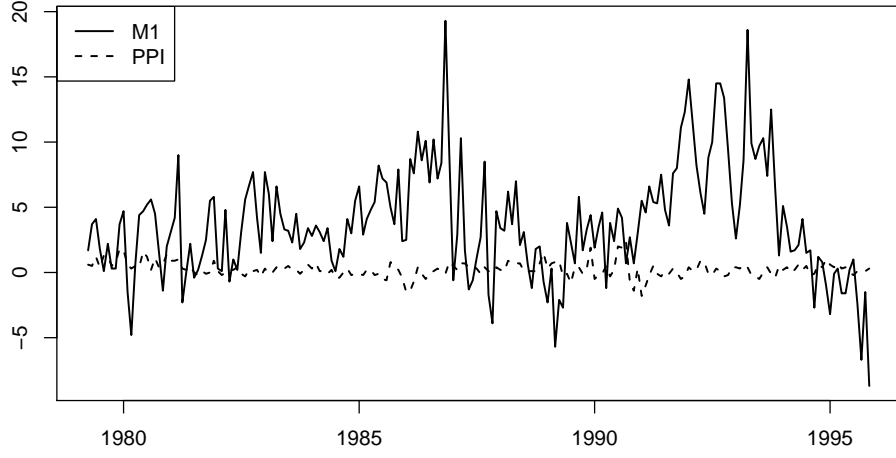


Figure 2: Evolution of the variations  $\Delta M1$  and  $\Delta PPI_{ACO}$ .

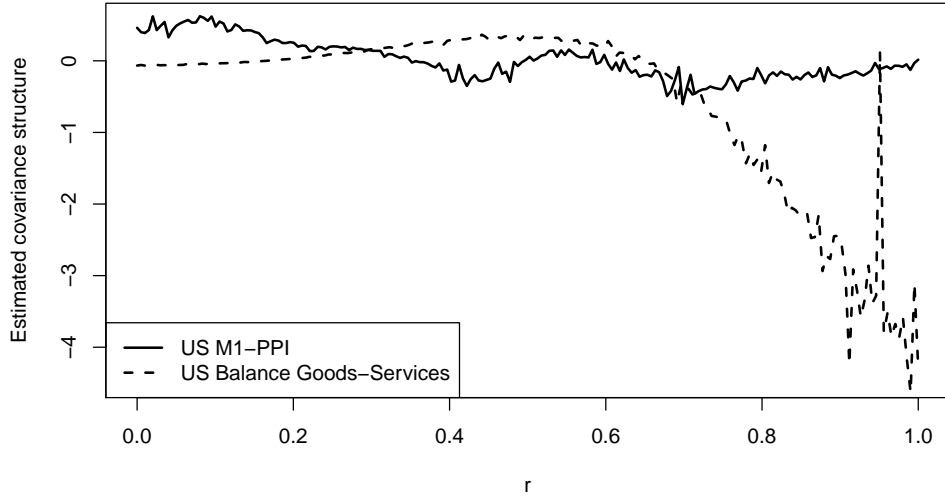


Figure 3: The Nadaraya-Watson kernel estimation of the covariance structure  $\Sigma^{12}(r)$  for the two data sets. The estimator is defined as in Patilea and Raïssi (2012) which showed that such estimator is consistent under **A1** unless at the break points.



	Number of lags		
	3	6	12
$BP_{OLS}$	0.4384 [1.1198]	0.8169 [1.9071]	0.8165 [3.2870]

Table 3: The  $p$ -values of the Box-Pierce test adapted to our non standard framework. The corresponding statistics are displayed into brackets. The  $BP_{OLS}$  corresponds to the portmanteau test based on the OLS proxies of the  $u_t$ 's.

$\hat{A}_{01}$
0.643 [0.064]    -1.124 [0.360]
-0.009 [0.007]    0.439 [0.102]

Table 4: The OLS estimators of the matrix  $A_{01}$  (see equation (2.1)) for the adjusted VAR(1) model. Standard deviations of the parameters are displayed into brackets.

	$W_{st}$	$W_w$	$W_b$
$p$ -values	0.268 [1.225]	0.201 [1.632]	0.058 [10.54]

Table 5: The  $p$ -values of the  $W_{st}$ ,  $W_w$  and  $W_b$  tests. The corresponding test statistics are displayed into brackets.

#### 4.2.2 Merchandise trade balance and balance on services in the U.S.A.

The merchandise trade balance and the balance on services can be seen as indicators of the economic health of a country. The U.S. merchandise trade balance is the account which redraws the value of the exported goods and the value of the imported goods. The U.S. balance on services is similarly the account which redraws the value of the exported services and the value of the imported services. Here, we search to quantify if it exists an instantaneous causality relation between these two macroeconomic indicators. The data are provided by the Bureau Analysis of the U.S. Department of Commerce and go from the 01/1960 to the 01/2011 with quarterly frequency. The length of the series is  $T = 204$ . They are available on the web site

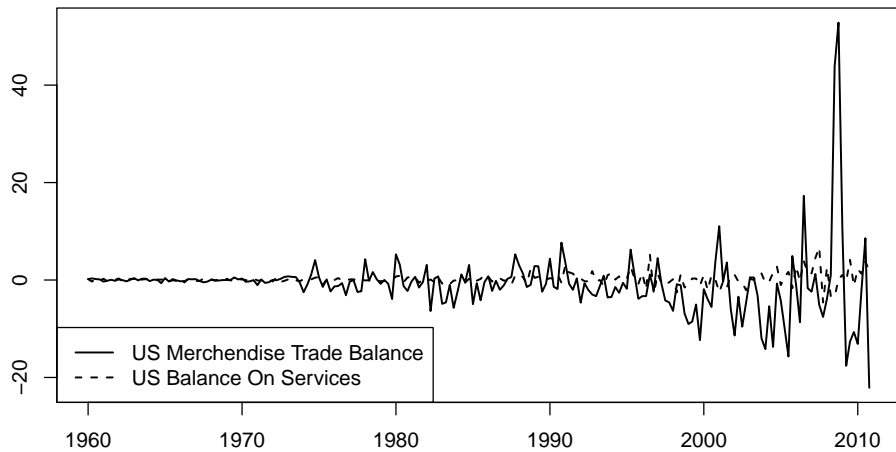


Figure 4: Evolution of the first differences of the U.S. merchandise trade balance and the U.S. balance on services in billion of U.S. dollars.

of the Federal Reserve Bank of St. Louis (Series ID : BOPBM and BOPBSV).

Similarly to the first data set, we consider the first differences of the data (see Figure 4). A VAR(2) model is adjusted to the data (estimation results not reported here). The adequacy of the model is again checked using portmanteau tests which are valid in our framework. The portmanteau test suggests to choose a VAR(2) model. Indeed the  $p$ -value of the  $BP_{OLS}$  test is 0.65[5.29] and for 5 autocorrelations in the portmanteau statistics (the portmanteau statistic is given into brackets). The  $p$ -values of the three tests are next computed from the residuals as for the first data set. The outcomes displayed in Table 6 show that the considered tests have quite different results. In view of the non constant variance of the studied series (see Figure 3), the result corresponding to the  $W_b$  test is more reliable.

	$W_{st}$	$W_w$	$W_b$
$p$ -value	0.0498 [3.848]	0.341 [0.907]	0.441 [190.142]

Table 6: The  $p$ -values of the  $W_{st}$ ,  $W_w$  and  $W_b$  tests. The corresponding test statistics are displayed into brackets.

## 5 Conclusion

In this paper we studied the problem of testing instantaneous causality in the important case where the unconditional variance is time-varying. The properties of the Wald tests based on the assumption of constant unconditional variance are investigated in our non standard framework. It emerges that this kind of tests may have no power in this non standard framework. As a consequence we proposed a new bootstrap test for testing the instantaneous causality hypothesis in the important case where the unconditional variance structure is time-varying. In particular we found that the proposed bootstrap test is consistent. We illustrated these theoretical results through a set of numerical experiments. The outcomes of macroeconomic data sets suggest that the Wald test may deliver results which are quite different from the bootstrap test. Although our non-standard framework allows for non constant variance, it assumes that the structural innovations cannot display conditional heteroscedasticity. This case could be the object of interesting further researches.

## Appendix

The following Lemmas are similar to Lemmas 7.2, 7.3 and 7.4 of Patilea and Raïssi (2012), so that the proofs are omitted. Introduce  $v_t = \text{vec}(u_{1t}u'_{2t} - \Sigma_t^{12})$  with  $u_t = (u'_{1t}, u'_{2t})'$  and recall that  $\vartheta_t = u_{2t} \otimes u_{1t}$ .

**Lemma 5.1.** *Under A1 we have*

$$\lim_{T \rightarrow \infty} E [v_{[Tr]} v'_{[Tr]}] = (G_2(r) \otimes G_1(r)) \mathcal{M}(G_2(r) \otimes G_1(r))' - \text{vec}(\Sigma^{12}(r)) \text{vec}(\Sigma^{12}(r))',$$

and

$$\lim_{T \rightarrow \infty} E [\vartheta_{[Tr]}] = \text{vec}(\Sigma^{12}(r))$$

for values  $r \in (0, 1]$  at which the functions  $g_{ij}(r)$  are continuous.

**Lemma 5.2.** *Under A1 we have*

$$T^{-1} \sum_{t=1}^T v_t v'_t \rightarrow \lim_{T \rightarrow \infty} T^{-1} \sum_{t=1}^T E(v_t v'_t), \quad (5.1)$$

and

$$T^{-1} \sum_{t=1}^T \vartheta_t \rightarrow \lim_{T \rightarrow \infty} T^{-1} \sum_{t=1}^T E(\vartheta_t).$$

**Lemma 5.3.** *Under A1 we have*

$$\begin{aligned} T^{-1} \sum_{t=1}^T E(v_t v'_t) &\rightarrow \int_0^1 (G_2(r) \otimes G_1(r)) \mathcal{M}(G_2(r) \otimes G_1(r))' dr \\ &\quad - \int_0^1 \text{vec}(\Sigma^{12}(r)) \text{vec}(\Sigma^{12}(r))' dr, \end{aligned} \quad (5.2)$$

and

$$T^{-1} \sum_{t=1}^T E(\vartheta_t) \rightarrow \int_0^1 \Sigma^{12}(r) dr.$$

**Proof of Lemma 2.1** We first give the proof of (2.3). Let us define  $u_t(\theta) = (u'_{1t}(\theta), u'_{2t}(\theta))' = X_t - (\tilde{X}_{t-1} \otimes I_d)\theta$  for any  $\theta \in \mathbb{R}^{pd^2}$ . From the Mean Value Theorem we have

$$\begin{aligned} T^{-1} \sum_{t=1}^T \hat{u}_{2t} \otimes \hat{u}_{1t} &= T^{-1} \sum_{t=1}^T u_{2t} \otimes u_{1t} \\ &\quad + T^{-1} \sum_{t=1}^T \left\{ \frac{\partial u_{2t}(\theta)}{\partial \theta'} \otimes u_{1t}(\theta) + u_{2t}(\theta) \otimes \frac{\partial u_{1t}(\theta)}{\partial \theta'} \right\}_{\theta=\theta^*} (\hat{\theta} - \theta_0), \end{aligned} \quad (5.3)$$

where  $\theta^*$  is between  $\hat{\theta}$  and  $\theta_0$ , and  $\partial u_t(\theta)/\partial \theta' = -\tilde{X}'_{t-1} \otimes I_d$  is uncorrelated with  $u_t$ . Hence we write  $T^{-\frac{1}{2}} \sum_{t=1}^T \hat{v}_t = T^{-\frac{1}{2}} \sum_{t=1}^T v_t + o_p(1)$  with  $v_t = \text{vec}(u_{1t}u'_{2t} - \Sigma_t^{12})$ , since the estimator  $\hat{\theta}$  is such that  $\sqrt{T}(\hat{\theta} - \theta_0) = O_p(1)$ . The process  $(v_t)$  is a martingale difference sequence, so that from the Lindeberg central limit theorem  $T^{-\frac{1}{2}} \sum_{t=1}^T v_t$  is asymptotically normal with mean zero. The expression of the covariance matrix  $\Omega$  can be obtained as follows, using  $E(\epsilon_t \epsilon'_t | \mathcal{F}_{t-1}) = I_d$  and Lemmas 5.1 and 5.3:

$$\begin{aligned}
\Omega &:= \lim_{T \rightarrow \infty} T^{-1} \text{Cov}\left(\sum_{t=1}^T v_t, \sum_{t=1}^T v_t\right) = \lim_{T \rightarrow \infty} T^{-1} \sum_{t=1}^T E(v_t v'_t) \\
&= \lim_{T \rightarrow \infty} T^{-1} \sum_{t=1}^T E \left\{ (u_{2t} \otimes u_{1t} - \text{vec}(\Sigma_t^{12})) (u_{2t} \otimes u_{1t} - \text{vec}(\Sigma_t^{12}))' \right\} \\
&= \lim_{T \rightarrow \infty} T^{-1} \sum_{t=1}^T [E \{ (u_{2t} u'_{2t}) \otimes (u_{1t} u'_{1t}) \} - \text{vec}(\Sigma_t^{12}) \text{vec}(\Sigma_t^{12})'] \quad (5.4) \\
&= \lim_{T \rightarrow \infty} T^{-1} \sum_{t=1}^T [E \{ (H_{2t} \epsilon_t \epsilon'_t H'_{2t}) \otimes (H_{1t} \epsilon_t \epsilon'_t H'_{1t}) \} - \text{vec}(\Sigma_t^{12}) \text{vec}(\Sigma_t^{12})'] \\
&= \lim_{T \rightarrow \infty} T^{-1} \sum_{t=1}^T [(H_{2t} \otimes H_{1t}) E(\epsilon_t \epsilon'_t \otimes \epsilon_t \epsilon'_t) (H_{2t} \otimes H_{1t})' - \text{vec}(\Sigma_t^{12}) \text{vec}(\Sigma_t^{12})'] \\
&= \int_0^1 (G_2(r) \otimes G_1(r)) \mathcal{M}(G_2(r) \otimes G_1(r))' dr - \int_0^1 \text{vec}(\Sigma^{12}(r)) \text{vec}(\Sigma^{12}(r))' dr,
\end{aligned}$$

where the identity  $(F \otimes J)(K \otimes L) = (FK) \otimes (JL)$  is used for matrices of appropriate dimensions. The proof of (2.2) follow directly from Lemmas 5.1, 5.2, 5.3 and equation (5.3).

For the proof of (2.4) again note that  $\hat{v}_t$  can be replaced by  $v_t$  from (5.3). We write

$$\begin{aligned}
v_t &= u_{2t} \otimes u_{1t} - \text{vec}(\Sigma_t^{12}) \\
&= H_{2t} \epsilon_t \otimes H_{1t} \epsilon_t - \text{vec}(H_{1t} H'_{2t}) \\
&= (H_{2t} \otimes H_{1t}) \{ \epsilon_t \otimes \epsilon_t - \text{vec}(I_d) \}.
\end{aligned}$$

Define  $v_t^\epsilon := \epsilon_t \otimes \epsilon_t - \text{vec}(I_d)$ . We have  $E(v_t^\epsilon) = 0$  and  $\text{Var}(v_t^\epsilon) = E(\epsilon_t \epsilon_t' \otimes \epsilon_t \epsilon_t') - \text{vec}(I_d)\text{vec}(I_d)' =: \tilde{\Omega}$ . Therefore from Theorem 3.1 of Hansen (1992) it follows that

$$T^{-\frac{1}{2}} \sum_{t=1}^{[Ts]} v_t \Rightarrow \int_0^s (G_2(r) \otimes G_1(r)) dB_{\tilde{\Omega}}(r)$$

for  $0 \leq s \leq 1$ .  $\square$

**Proof of (3.7)** For the sake of simplicity and with no loss of generality (see (5.3)) let us assume that  $X_t = u_t$ , so that the error process is observed and there is no autoregressive parameters to estimate. Conditionally on the  $u_t$ 's,  $\delta_s^{(i)}$  is a Gaussian process with independent increments and variance

$$E^*(\delta_s^{(i)} \delta_s^{(i)'}) = T^{-1} \sum_{t=1}^{[Ts]} \vartheta_t \vartheta_t',$$

where  $E^*(.)$  is the expectation under the bootstrap probability measure. The result follows if

$$T^{-1} \sum_{t=1}^{[Ts]} \vartheta_t \vartheta_t' \rightarrow \int_0^s (G_2(r) \otimes G_1(r)) \mathcal{M}(G_2(r) \otimes G_1(r))' dr,$$

uniformly for all  $s \in [0, 1]$ . Since  $T^{-1} \sum_{t=1}^{[Ts]} \vartheta_t \vartheta_t'$  is monotonically increasing and the limit function is continuous, it suffices to establish the pointwise convergence following Hansen (2000, proof of Lemma A.10). This holds using similar arguments as for (5.1) and (5.2), see Patilea and Raïssi (2012) Lemmas 7.3 and 7.4.  $\square$

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